

6-Body Central Configurations Formed by Two Isosceles Triangles*

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Abstract: In this paper, we show the existence of a class of 6-body central configurations with two isosceles triangles; which are congruent to each other and keep some distance. We also study the necessary conditions about masses for the bodies which can form a central configuration.

Keywords : 6-body problems, central configurations, isosceles triangles.

MSC: 34C15, 34C25.

1 Introduction and Main Results

The Newtonian N-body problem concerns the motion of N particles with masses $m_j \in R^+$ and positions $q_j \in R^3 (j = 1, 2, \dots, N)$, the motion is governed by Newton's second law and the Universal law:

$$m_j \ddot{q}_j = -\frac{\partial U(q)}{\partial q_j}, \quad (1.1)$$

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where $q = (q_1, q_2, \dots, q_N)$ and $U(q)$ is Newtonian potential:

$$U(q) = \sum_{1 \leq j < k \leq N} \frac{m_j m_k}{|q_j - q_k|}, \quad (1.2)$$

Consider the space

$$X = \{q = (q_1, q_2, \dots, q_N) \in R^{3N} : \sum_{j=1}^N m_j q_j = 0\}, \quad (1.3)$$

i.e., suppose that the center of mass is fixed at the origin of the space. Because the potential is singular when two particles have same position, it is natural to assume that the configuration avoids the collision set $\Delta = \{q = (q_1, \dots, q_N) : q_j = q_k \text{ for some } k \neq j\}$. The set $X \setminus \Delta$ is called the configuration space.

Definition 1.1 ([17, 22]): A configuration $q = (q_1, q_2, \dots, q_N) \in X \setminus \Delta$ is called a central configuration if there exists a constant λ such that

$$\sum_{j=1, j \neq k}^N \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, \quad 1 \leq k \leq N. \quad (1.4)$$

The value of constant λ in (1.4) is uniquely determined by

$$\lambda = \frac{U}{I}, \quad (1.5)$$

where

$$I = \sum_{k=1}^N m_k |q_k|^2. \quad (1.6)$$

Since the general solution of the N-body problem can't be given, great importance has been attached to search for particular solutions from the very beginning. A homographic solution is that a configuration is preserved for all time. Central configurations and homographic solutions are

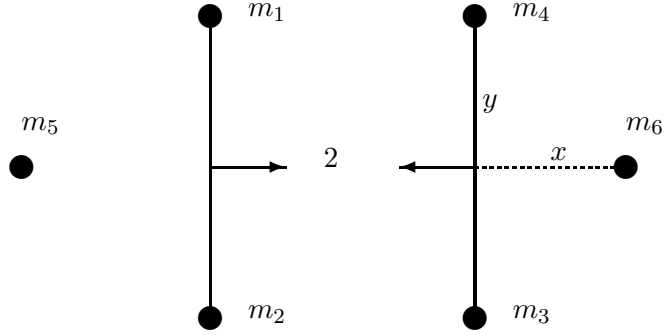
linked by the Laplace theorem ([17,22]). Collapse orbits and parabolic orbits have relations with the central configurations([15,16]). So finding central configurations becomes very important. The main general open problem for the central configurations is due to Winter[22] and Smale[20]: Is the number of planar central configurations finite for any choice of positive masses m_1, \dots, m_N ? Hampton and Moeckel([6]) have proved this conjecture for four any given positive masses.

In 1941, Wintner([22]) have studied regular polygon central configurations. Moeckel ([11]), Zhang and Zhou([23]) have studied highly symmetrical central configuration of Newtonian N-body problems. Llibre and Mello ([8]) have studied a class of 6-body central configurations.

Based the above works, we find a classes of central configurations in the 6-body problems, for which three bodies are at the vertices of an isosceles triangles, the others are located at the vertices of another isosceles triangles and the two triangles are congruent to each other;

Related assumptions will be interpreted more precisely in the following.

Assume m_1, m_2 and m_5 are located the vertices of a isosceles triangles Δ_1 ; m_3, m_4 and m_6 are located at the vertices of another isosceles triangles Δ_2 . Δ_1 and Δ_2 are coplanar and are congruent to each other; $q_1 - q_2$ is parallel to $q_4 - q_3$; $|q_1 - q_4| < |q_5 - q_6|$; q_5 and q_6 are located at the common perpendicular bisector for q_1q_2 and q_3q_4 . Without loss of generality we can take a coordinate system such that $q_1 = (-1, y), q_2 = (-1, -y), q_3 = (1, -y), q_4 = (1, y), q_5 = (-1 - x, 0), q_6 = (1 + x, 0)$. (See Fig).



We have:

Theorem 1.1: If m_1, m_2, m_3, m_4, m_5 and m_6 form a central configuration, then $m_1 = m_2 = m_3 = m_4$ and $m_5 = m_6$.

Theorem 1.2: Assume $m_1 = m_2 = m_3 = m_4 = 1, m_5 = m_6 = m$, then there exists a non-empty open set $U \subset (1, +\infty)$, $\varphi(y) \in C(U)$ such that $\varphi(\sqrt{3}) = 1$ and $m = m(x, y) = m(\varphi(y), y)$, so that $(q_1, q_2, q_3, q_4, q_5, q_6)$ form a central configuration.

Remark: When $x = 1$ and $y = \sqrt{3}$, q_i is the vertex of a regular 6-gons ($i = 1, \dots, 6$).

2 The Proofs of Theorems

2.1 The Proof of Theorem 1.1

Note that

$$\begin{aligned}
 \sum_{j=1, j \neq k}^N \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) &= -\lambda m_k q_k = -\lambda m_k (q_k - 0) \\
 &= -\lambda m_k \left(q_k - \frac{\sum_{j=1}^N m_j q_j}{M} \right) = -m_k \frac{\lambda}{M} \sum_{j=1}^N m_j (q_k - q_j)
 \end{aligned} \tag{2.1}$$

where $M = \sum_{i=1}^N m_i$.

So (1.4) is also equivalent to

$$\sum_{j=1, j \neq k}^N m_j \left(\frac{1}{|q_j - q_k|^3} - \frac{\lambda}{M} \right) (q_j - q_k) = 0 \quad (2.2)$$

By (2.2) we have

$$\begin{aligned} & 0m_1 + 0m_2 + 2\left(\frac{1}{|4 + 4y^2|^{3/2}} - \frac{\lambda}{M}\right)m_3 + 2\left(\frac{1}{2^3} - \frac{\lambda}{M}\right)m_4 \\ & -x\left(\frac{1}{|x^2 + y^2|^{3/2}} - \frac{\lambda}{M}\right)m_5 + (2+x)\left(\frac{1}{|x^2 + y^2 + 4x + 4|^{3/2}} - \frac{\lambda}{M}\right)m_6 = 0, \end{aligned} \quad (2.3)$$

$$\begin{aligned} & 0m_1 - 2y\left(\frac{1}{|2y|^3} - \frac{\lambda}{M}\right)m_2 - 2y\left(\frac{1}{|4 + 4y^2|^{3/2}} - \frac{\lambda}{M}\right)m_3 + 0m_4 \\ & -y\left(\frac{1}{|x^2 + y^2|^{3/2}} - \frac{\lambda}{M}\right)m_5 - y\left(\frac{1}{|x^2 + y^2 + 4x + 4|^{3/2}} - \frac{\lambda}{M}\right)m_6 = 0, \end{aligned} \quad (2.4)$$

$$\begin{aligned} & 0m_1 + 0m_2 + 2\left(\frac{1}{2^3} - \frac{\lambda}{M}\right)m_3 + 2\left(\frac{1}{|4 + 4y^2|^{3/2}} - \frac{\lambda}{M}\right)m_4 \\ & -x\left(\frac{1}{|x^2 + y^2|^{3/2}} - \frac{\lambda}{M}\right)m_5 + (2+x)\left(\frac{1}{|x^2 + y^2 + 4x + 4|^{3/2}} - \frac{\lambda}{M}\right)m_6 = 0, \end{aligned} \quad (2.5)$$

$$\begin{aligned} & 2y\left(\frac{1}{|2y|^3} - \frac{\lambda}{M}\right)m_1 + 0m_2 + 0m_3 + 2y\left(\frac{1}{|4 + 4y^2|^{3/2}} - \frac{\lambda}{M}\right)m_4 \\ & +y\left(\frac{1}{|x^2 + y^2|^{3/2}} - \frac{\lambda}{M}\right)m_5 + y\left(\frac{1}{|x^2 + y^2 + 4x + 4|^{3/2}} - \frac{\lambda}{M}\right)m_6 = 0, \end{aligned} \quad (2.6)$$

$$\begin{aligned} & -2\left(\frac{1}{|4 + 4y^2|^{3/2}} - \frac{\lambda}{M}\right)m_1 - 2\left(\frac{1}{2^3} - \frac{\lambda}{M}\right)m_2 + 0m_3 + 0m_4 \\ & -(2+x)\left(\frac{1}{|x^2 + y^2 + 4x + 4|^{3/2}} - \frac{\lambda}{M}\right)m_5 + x\left(\frac{1}{|x^2 + y^2|^{3/2}} - \frac{\lambda}{M}\right)m_6 = 0, \end{aligned} \quad (2.7)$$

$$\begin{aligned} & 2y\left(\frac{1}{|4 + 4y^2|^{3/2}} - \frac{\lambda}{M}\right)m_1 + 0m_2 + 0m_3 + 2y\left(\frac{1}{|2y|^3} - \frac{\lambda}{M}\right)m_4 \\ & +y\left(\frac{1}{|x^2 + y^2 + 4x + 4|^{3/2}} - \frac{\lambda}{M}\right)m_5 + y\left(\frac{1}{|x^2 + y^2|^{3/2}} - \frac{\lambda}{M}\right)m_6 = 0, \end{aligned} \quad (2.8)$$

$$\begin{aligned}
& -2\left(\frac{1}{2^3} - \frac{\lambda}{M}\right)m_1 - 2\left(\frac{1}{|4+4y^2|^{3/2}} - \frac{\lambda}{M}\right)m_2 + 0m_3 + 0m_4 \\
& - (2+x)\left(\frac{1}{|x^2+y^2+4x+4|^{3/2}} - \frac{\lambda}{M}\right)m_5 + x\left(\frac{1}{|x^2+y^2|^{3/2}} - \frac{\lambda}{M}\right)m_6 = 0,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
& 0m_1 - 2y\left(\frac{1}{|4+4y^2|^{3/2}} - \frac{\lambda}{M}\right)m_2 - 2y\left(\frac{1}{|2y|^3} - \frac{\lambda}{M}\right)m_3 + 0m_4 \\
& - y\left(\frac{1}{|x^2+y^2+4x+4|^{3/2}} - \frac{\lambda}{M}\right)m_5 - y\left(\frac{1}{|x^2+y^2|^{3/2}} - \frac{\lambda}{M}\right)m_6 = 0,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
& x\left(\frac{1}{|x^2+y^2|^{3/2}} - \frac{\lambda}{M}\right)(m_1+m_2) + \\
& (x+2)\left(\frac{1}{|x^2+y^2+4x+4|^{3/2}} - \frac{\lambda}{M}\right)(m_3+m_4) \\
& + 0m_5 + 2(1+x)\left(\frac{1}{|2(1+x)|^3} - \frac{\lambda}{M}\right)m_6 = 0,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
& (x+2)\left(\frac{1}{|x^2+y^2+4x+4|^{3/2}} - \frac{\lambda}{M}\right)(m_1+m_2) \\
& + x\left(\frac{1}{|x^2+y^2|^{3/2}} - \frac{\lambda}{M}\right)(m_3+m_4) + \\
& + 2(1+x)\left(\frac{1}{|2(1+x)|^3} - \frac{\lambda}{M}\right)m_5 + 0m_6 = 0,
\end{aligned} \tag{2.12}$$

By(2.3),(2.5),(2.7) and (2.9),we have:

$$\begin{aligned}
& (m_3 - m_4)\left(\frac{1}{|4+4y^2|^{3/2}} - \frac{1}{2^3}\right) = 0, \\
& (m_1 - m_2)\left(\frac{1}{|4+4y^2|^{3/2}} - \frac{1}{2^3}\right) = 0.
\end{aligned} \tag{2.13}$$

By(2.4),(2.6),(2.8) and (2.10),we have:

$$\begin{aligned}
& \left(\frac{1}{|4+4y^2|^{3/2}} - \frac{1}{2^3}\right)(m_1 - m_3) + \left(\frac{1}{|2y|^3} - \frac{1}{2^3}\right)(m_4 - m_2) = 0, \\
& \left(\frac{1}{|2y|^3} - \frac{1}{2^3}\right)(m_1 - m_3) + \left(\frac{1}{|4+4y^2|^{3/2}} - \frac{1}{2^3}\right)(m_4 - m_2) = 0.
\end{aligned} \tag{2.14}$$

By(2.13) and (2.14),we have:

$$m_1 = m_2 = m_3 = m_4. \tag{2.15}$$

By (2.4),(2.6),(2.11), (2.12) and (2.15),we have

$$m_5 = m_6. \quad (2.16)$$

The proof of **Theorem1.1** is completed.

2.2 The Proof of Theorem 1.2

Notice that (q_1, \dots, q_6) is a central configuration if and only if

$$\sum_{j=1, j \neq k}^6 \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, 1 \leq k \leq 6. \quad (2.17)$$

Since the symmetries,(2.17) is equivalent to

$$\sum_{j=1, j \neq k}^6 \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, k = 2, 5. \quad (2.18)$$

Now (2.18) is equivalent to

$$\lambda = \frac{1}{4} + \frac{1}{4|1 + y^2|^{3/2}} - \frac{xm}{|x^2 + y^2|^{3/2}} + \frac{(2 + x)m}{|x^2 + y^2 + 4x + 4|^{3/2}}, \quad (2.19)$$

$$\lambda = \frac{1}{4y^3} + \frac{1}{4|1 + y^2|^{3/2}} + \frac{m}{|x^2 + y^2|^{3/2}} + \frac{m}{|x^2 + y^2 + 4x + 4|^{3/2}}, \quad (2.20)$$

$$\lambda = \frac{2x}{|x^2 + y^2|^{3/2}(1 + x)} + \frac{2(2 + x)}{|x^2 + y^2 + 4x + 4|^{3/2}(1 + x)} + \frac{m}{4|1 + x|^3}, \quad (2.21)$$

(2.19) ,(2.20) and (2.21) are equivalent to

$$\left(\frac{1 + x}{|x^2 + y^2|^{3/2}} - \frac{1 + x}{|x^2 + y^2 + 4x + 4|^{3/2}} \right) m = \frac{1}{4} \left(1 - \frac{1}{y^3} \right), \quad (2.22)$$

$$\begin{aligned} & \left(\frac{1}{|x^2 + y^2|^{3/2}} + \frac{1}{|x^2 + y^2 + 4x + 4|^{3/2}} - \frac{1}{4|1 + x|^3} \right) m = \\ & \frac{2x}{|x^2 + y^2|^{3/2}(1 + x)} + \frac{2(2 + x)}{|x^2 + y^2 + 4x + 4|^{3/2}(1 + x)} - \frac{1}{4y^3} - \frac{1}{4|1 + y^2|^{3/2}}, \end{aligned} \quad (2.23)$$

By (2.22) we have

$$m = m_1(x, y) = \frac{1}{4} \left(1 - \frac{1}{y^3}\right) \frac{|x^2 + y^2|^{3/2} |x^2 + y^2 + 4x + 4|^{3/2}}{(1+x)(|x^2 + y^2 + 4x + 4|^{3/2} - |x^2 + y^2|^{3/2})} \quad (2.24)$$

$m_1(x, y) > 0$ if and only if $y > 1$.

By (2.23) we have

$$m = m_2(x, y) = \left[\frac{2x}{|x^2 + y^2|^{3/2}(1+x)} + \frac{2(2+x)}{|x^2 + y^2 + 4x + 4|^{3/2}(1+x)} - \frac{1}{4y^3} - \frac{1}{4|1 + y^2|^{3/2}} \right] \times \left[\frac{1}{|x^2 + y^2|^{3/2}} + \frac{1}{|x^2 + y^2 + 4x + 4|^{3/2}} - \frac{1}{4|1+x|^3} \right]^{-1} \quad (2.25)$$

Then (q_1, \dots, q_6) is a central configuration if and only if

$$m_1(x, y) = m_2(x, y) > 0 \quad (2.26)$$

It is obvious that

$$m_1(1, \sqrt{3}) = m_2(1, \sqrt{3}) = 1. \quad (2.27)$$

$$\frac{\partial m_1(1, \sqrt{3})}{\partial x} = \frac{1}{4}. \quad (2.28)$$

$$\frac{\partial m_2(1, \sqrt{3})}{\partial x} = \frac{1}{2} \frac{(9 - 16\sqrt{3})}{(27 + 4\sqrt{3})} \neq \frac{1}{4}. \quad (2.29)$$

By implicit function theorem, there exists a non-empty open set U and $\varphi(y) \in C(U)$ such that, $\sqrt{3} \in U$, $\varphi(\sqrt{3}) = 1$ and $\forall y \in U$, $m_1(\varphi(y), y) = m_2(\varphi(y), y)$.

The proof of **Theorem 1.2** is completed.

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